Theoretical study on a measuring method of rail axial stress via vibration modes of periodic track

Kazuhisa Abe*, Saki Shimizu*, Akira Aikawa**, Kazuhiro Koro*
* Niigata University
8050 Igarashi 2-nocho, Nishi-ku, Niigata, 950-2181, Japan
** Railway Technical Research Institute
2-8-38 Hikari-cho, Kokubunji-shi, Tokyo, 185-8540, Japan

SUMMARY
A measuring method of the rail axial stress is discussed based on a numerical model, in the context of the estimation by way of the resonant frequency. To represent the dynamic behavior of an infinite track, the periodicity in the structure is taken into account. By virtue of the Floquet’s theorem the equation of a unit cell which is defined by the sleeper spacing leads to an eigenvalue problem with respect to the frequency. We can then obtain the relation between the frequency and the axial load by extracting standing wave modes from the dispersion surfaces. The influence of material constants such as the stiffness of pads on the modes is also examined. The feasibility of the proposed method is validated through response analyses for a harmonic load. Furthermore, the influence of the stochasticity in the sleeper spacing on the vibration resonance is investigated, and an enhancement of the reliability of the measuring method is discussed.

Key Words: Rail Axial Stress, Numerical Analysis, Wave modes

1. INTRODUCTION
The employment of continuous welded rails (CWR) contributes to the improvement of the passenger comfort and the reduction of vibration and noise. In spite of this advantage, the CWR has a serious issue that the track may suffer buckling or breaks due to the temperature stress. Therefore, it is very important to monitor the axial load of rails in the context of the maintenance of railway.

Since effective management of the axial load results in the reduction of maintenance costs and the enhancement of safety, in the past few decades a number of measuring methods have been proposed [1]. However many of those methods cannot measure the absolute stress at any position. Although the vibration method which can measure the absolute axial load was investigated in Europe and America in the 1980’s, it has not been realized. It is thought that one of the main causes of this is the impropriety of the employed mathematical model which cannot capture the dynamic behavior of a periodic track.

Since the vibration method does not need the initial condition, it enables us to measure the axial load at anytime, anywhere. Consequently, this method will contribute to the saving of maintenance cost and the improvement of safety. In this study the capability of the method is discussed based on a numerical model. The railway track model is composed of infinite rails under an axial load and equidistantly
distributed supports. To represent the dynamic behavior of the infinite track, the periodicity in the structure is taken into account. It can be achieved by describing the equation of motion of an irreducible sub-structure called a unit cell. By virtue of the Floquet's theorem [2] the equation of the unit cell leads to an eigenvalue problem with respect to the frequency [3]. The eigenmodes exhibit the dispersion surfaces of propagating wave modes in the axial load-wavenumber-frequency space. The dynamic reaction due to harmonic or impulse loading will be dominated by standing wave modes. We can thus obtain the relation between the frequency and the axial load by extracting such modes from the dispersion surfaces. That is, the consideration of periodicity will help us establish a new method in which the axial load is measured by way of the predominant frequencies.

The feasibility of the proposed method is validated through response analyses for a harmonic load. The influence of mechanical parameters such as stiffness of pads and rail fastener on the modes is also examined.

As mentioned above, a railway track can be modeled by a periodic structure. This feature will make it possible to reveal the theoretical relation between the resonant frequency and the axial stress. However, in general, a track has some imperfection such as the randomness in the sleeper spacing. This perturbation may affect the dynamic response. Therefore, in this study the influence of the stochasticity in the sleeper spacing on the vibration resonance is investigated. Then an enhancement of the reliability of the measuring method is discussed.

2. DISPERSION ANALYSIS

2.1 Modeling of Railway Track

A railway track can be represented as a periodic structure characterized by the sleeper spacing $L$. A dynamic problem of this infinite system can then be reduced to that of an irreducible substructure called a unit cell as shown in Fig.1. In this study a track model consisting of rails, a sleeper and a number of springs is taken into account. The rails and sleeper are given by three-dimensional Timoshenko beams. Those are discretized by beam elements. In order for the dispersion analysis to capture wave modes propagating in the infinite periodic track, any dissipation is not considered.

2.2 Theory

The equation of motion of a unit cell is given by

$$\left[ \bar{W}^T \right] \left[ K - N C - \omega^2 M \right] \{ U \} = \left[ \bar{W}^T \right] \{ F \},$$

(1)
where \( \{W\} \) is a virtual displacement vector, \( \{U\} \) and \( \{F\} \) are nodal displacement and force vectors. \( \overline{\cdot} \) stands for a conjugate vector. \( [K] \) and \( [M] \) are the stiffness and mass matrices, respectively. \( \omega \) is the circular frequency, \( N \) is the axial load in the rail. \( [C] \) is a matrix concerning the axial stress.

By virtue of the Floquet’s principle a steady state solution satisfies the following relation [4]

\[
\mathbf{u}_L = \mathbf{u}_0 e^{-i\kappa L}, \quad \mathbf{f}_L = -\mathbf{f}_0 e^{-i\kappa L},
\]

(2)

where \( \mathbf{u}_0 \) and \( \mathbf{u}_L \) are nodal displacements at both ends of \( x=0 \) and \( x=L \). \( \mathbf{f}_0 \) and \( \mathbf{f}_L \) are nodal internal force vectors at these points. \( \kappa \) is the Floquet wavenumber.

Eqs.(1) and (2) lead to an eigenvalue problem as

\[
[K' - NC'] \{U'\} = \omega^2 [M'] \{U'\},
\]

(3)

where \( (\cdot)' \) denotes that the matrices and displacement vector have been degenerated due to the relation of eq.(2). Notice that components of the Hermitian matrices \( [K'] \), \( [C'] \) and \( [M'] \) are given by functions of the Floquet wavenumber \( \kappa \). Therefore, the eigenvalue problem of eq.(3) yields a relation of \( \kappa \), \( \omega \) and \( N \).

2.3 Dispersion Curves

In the analysis JIS 50kgN rails are considered with the sleeper spacing of \( L=0.6m \). Other analytical conditions are summarized in Tables 1, 2 and 3. Fig.2 is showing the relation between the Floquet wavenumber [4] and the frequency (Hz) for \( N=0, 2, 4, 6 \) (MN). Although, in general, the maximal rail load will be at most 1MN, to emphasize the effect of the axial load, the results for larger \( N \) are shown. In

![Dispersion Curves](image)

![Vibration modes](image)
the figure the horizontal lines correspond to vibration modes of the sleeper. From the figure we can see that these modes are independent of both the wavenumber and the axial load.

Wave modes (A, B, C, D, E) at \( \kappa L = \pi \) (rad) are given by standing waves which are locating at the boundary between the pass and stop bands [4, 5] and sensitive to \( N \). Since the standing wave modes can be easily excited by some loading pattern, it is expected that they contribute to the measurement of the axial stress. The displacement of each mode is shown in Fig.3. Their wave lengths are \( 2L \). The nodes of deflection for A, C and D are locating at the sleeper support, and those for B and E are locating at the mid-span. Although A and C seem to be the same mode, the former is dominated by the railhead displacement, while the latter is characterized by the rail foot displacement.

<table>
<thead>
<tr>
<th>Table 1 Parameters of JIS 50KgN rail</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mass density (Kg/m(^3))</td>
</tr>
<tr>
<td>Cross-sectional area (m(^2))</td>
</tr>
<tr>
<td>Young’s modulus (GPa)</td>
</tr>
<tr>
<td>Poisson’s ratio</td>
</tr>
<tr>
<td>Geometrical moment of inertia (Y-axis) (m(^4))</td>
</tr>
<tr>
<td>Geometrical moment of inertia (Z-axis) (m(^4))</td>
</tr>
<tr>
<td>Shear factor (Y-axis)</td>
</tr>
<tr>
<td>Shear factor (Z-axis)</td>
</tr>
<tr>
<td>St Venant torsional constant (m(^4))</td>
</tr>
<tr>
<td>Distance between centroid and rail foot (m)</td>
</tr>
<tr>
<td>Distance between centroid and shear center (m)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Table 2 Parameters of sleeper</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mass density (Kg/m(^3))</td>
</tr>
<tr>
<td>Cross-sectional area (m(^2))</td>
</tr>
<tr>
<td>Length (m)</td>
</tr>
<tr>
<td>Young’s modulus (GPa)</td>
</tr>
<tr>
<td>Poisson’s ratio</td>
</tr>
<tr>
<td>Geometrical moment of inertia (Y-axis) (m(^4))</td>
</tr>
<tr>
<td>Geometrical moment of inertia (Z-axis) (m(^4))</td>
</tr>
<tr>
<td>Shear factor (Y-axis)</td>
</tr>
<tr>
<td>Shear factor (Z-axis)</td>
</tr>
<tr>
<td>St Venant torsional constant (m(^4))</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Table 3 Spring constants</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rail fastener</td>
</tr>
<tr>
<td>( k_{xx} ) (MN/m)</td>
</tr>
<tr>
<td>( k_{yy} ) (MN/m)</td>
</tr>
<tr>
<td>( k_{zz} ) (MN/m)</td>
</tr>
<tr>
<td>( k_{cx} ) (MN·m/rad)</td>
</tr>
<tr>
<td>( k_{cy} ) (MN·m/rad)</td>
</tr>
<tr>
<td>( k_{cz} ) (MN·m/rad)</td>
</tr>
<tr>
<td>Sleeper pad, ballast</td>
</tr>
<tr>
<td>( k_{sx} ) (MN/m)</td>
</tr>
<tr>
<td>( k_{sy} ) (MN/m)</td>
</tr>
<tr>
<td>( k_{sz} ) (MN/m)</td>
</tr>
<tr>
<td>( k_{cs} ) (MN·m/rad)</td>
</tr>
</tbody>
</table>
2.4 Influence of Spring Constants

Parameters such as the rail pad stiffness and the ballast stiffness have some randomness. In the following, the sensitivity of modes to spring constants at the rail support is investigated. Fig. 4 is showing the influence of spring constants in $x$, $y$ and $z$ directions. The influence of rotational springs is also shown in Fig. 5. Since the influence of the ballast stiffness is very small, the result is not shown. From these figures it is found that modes A and C are almost independent of the stiffness. Notice that these two modes have no horizontal displacement at the rail fastener but have rotation with respect to $z$ and $x$ axes due to the deflection and torsion. In spite of this, the effect of the rotational springs is negligible.

From Figs. 4 and 5 the sensitivities of the natural frequencies to the axial stress at $N=0$ are 10Hz/MN and 5Hz/MN for modes A and C, respectively. Therefore, it can be concluded that mode A is suitable for the axial load measurement.

![Fig. 4 Influence of translational spring constants on frequency-axial load relations](image1)

![Fig. 5 Influence of rotational spring constants on frequency-axial load relations](image2)

3. RESPONSE ANALYSIS OF A TRACK SUBJECTED TO A HARMONIC LOAD

In the prior section the relation between the axial load and the natural frequency has been investigated. In order to extract this relation by a measurement, a resonance mode should be excited. In practice, it can be achieved by harmonic or impulse loads. Although the wave modes of a periodic track extend for its infinite region, the response to a point load is to be a localized resonance. Therefore, the standing wave modes such as mode A cannot be reproduced satisfactorily by loading applied to a portion of the track. Because of this, it is worthwhile to validate the feasibility of the proposed measuring method through a numerical simulation of the track loading. For this purpose the response to a harmonic load is analyzed. A three-dimensional beam model as depicted in Fig. 1 is employed. A finite but long track
model with damping is assembled by connecting 400 unit cells. A horizontal-harmonic load acting at a mid-span is considered. In order to excite the response corresponding to mode A, in which the vibration is characterized by the lateral swing of the rail, the load is applied at the railhead as illustrated in Fig.6. The horizontal displacement at the top of the rail is evaluated. The relation between the resonant frequency and the axial load is shown in Fig.7 with the result of mode A (Fig.4). From a good agreement between these results the utility of mode A can be evidenced, and it can be excited by the harmonic loading. Although the small discrepancy between the resonant frequency and mode A may induce the misestimate of about 50kN, it will be allowable.

4. INFLUENCE OF RANDOMNESS IN SLEEPER SPACING ON RESONANT FREQUENCY

In the previous section the relation between the rail axial load and the resonant frequencies was investigated based on a periodic track model. However, in general, a track has some randomness disturbing the periodicity. Therefore, in the following, we shall discuss the influence of the stochasticity in the sleeper spacing on the vibration resonance. To achieve this, a one-dimensional track model is employed under the consideration of the random sleeper spacing. The effect of the randomness in a track has been studied by Wu and Thompson [6] and Oscarsson [7]. In Ref.[6] the sleeper spacing and the ballast stiffness are considered as random variables. It was concluded that the former may affect the pinned-pinned resonance. In Ref.[7] the stochasticity in a track was discussed by means of the perturbation method in which the first-order perturbation was taken into account. However, as will be
shown in this section, the second-order perturbation is essential to represent the influence of the sleeper spacing on the resonant frequency.

4.1 Relation between Sleeper Position and Resonant Frequency

Fig. 8 shows the relation between the deviation in the position of a sleeper from the periodicity and the predominant resonant frequency of the rail. Since the objective of this section is to reveal the influence of the randomness in the sleeper spacing on the dynamic response, for the sake of simplicity, only the horizontal displacement is considered under a one-dimensional beam model subjected to no axial load. In the figure the difference in the resonant frequency $\beta$ (Hz) from that of the periodic track is shown as a function of the deviation $\varepsilon_n$ in the $n$th ($n=10, 20, 40$) sleeper location. Here the first sleeper is assigned to the right end of the loading span. The sleepers are numbered from the first one to rightwards. The figure shows that $\beta$ is governed by the second-order terms of $\varepsilon_i$.

Through numerical experiments, it was found that the response tends to possess two resonant frequencies for larger $\varepsilon$. One is decreases with increasing $\varepsilon$, the other stays around $\beta=0$ for any $\varepsilon$. The latter resonance becomes the predominance at the larger $\varepsilon$. The discontinuities in the $\varepsilon$-$\beta$ curves are caused by this reversal. The further the distance from a sleeper, the smaller the deviation $\varepsilon$ giving the discontinuity. From these results, $\beta$ can be approximated by a generalized form as

$$\beta := f(\varepsilon) - f(0) = \sum_{i,j} A_{ij}(\varepsilon) \varepsilon_i \varepsilon_j,$$

where $f(\varepsilon)$ is the dominant resonant frequency under an imperfection $\varepsilon$, and $A_{ij}(\varepsilon) \approx 0$ for large $\varepsilon$, and $\varepsilon_j$.

4.2 Evaluation of Stochasticity

Let us assume that $\varepsilon_i$ is given by a random variable characterized by a Gaussian distribution with mean value of $E(\varepsilon)=0$. The smaller the standard deviation $\sigma(\varepsilon)$, the lower the probability for a large $\varepsilon$ becomes. Therefore, $A_{ij}(\varepsilon)$ can approximately be replaced by a constant coefficient irrespective of the value of $\varepsilon$ for a small $\sigma(\varepsilon)$. In this case, $E(\beta)$ will be expressed by

$$E(\beta) = \sum_i A_{ii} \cdot \sigma^2(\varepsilon).$$

Fig. 9 shows the relation between $\sigma^2(\varepsilon)$ and $E(\beta)$, where the expectation of $\beta$ is evaluated by 1000 random tracks generated for each $\sigma^2(\varepsilon)$. From the figure it can be confirmed that $E(\beta)$ is proportional to $\sigma^2(\varepsilon)$ within the range of $\sigma^2(\varepsilon)<0.0002$. In practice, this range will cover almost every track in Japan. The breakage of the proportionality will be caused by the occurrence of the discontinuity in $\beta(\varepsilon)$ for large $\varepsilon$.

From eqs. (4) and (5) we obtain the following relation,

$$f(0) = E(f(\varepsilon)) - \sum_i A_{ii} \cdot \sigma^2(\varepsilon).$$

$E(f(\varepsilon))$ can be estimated by measuring the resonant frequency for sampling spans around the site. $\sigma(\varepsilon)$ can also be evaluated by way of a set of statistical measurements. $\Sigma A_{ij}$ is to be obtained through numerical experiments as shown in Fig. 9. These data will enable us to correct the influence of the
stochasticity in the sleeper spacing by using eq.(6).

Through further analyses, it has also been found that the influence of the axial load on the statistical features is negligible.

![Fig.8 Relation between deviation of sleeper position and resonant frequency](image1)

![Fig.9 Relation between $\sigma^2(\epsilon)$ and $E(\beta)$](image2)

5. CONCLUSION

The feasibility of a measuring method for the rail axial stress has been investigated by means of numerical analyses. The relation of axial load-wavenumber-frequency has been obtained through the dispersion analysis for wave modes propagating in a track. It is concluded that the standing wave mode dominated by horizontal deflection accompanied with torsion is suitable for the identification of the axial load. Since this mode has nodes at the sleeper supports, it is almost independent of parameters such as the stiffness of the fastener. Besides, rather high sensitivity in the resonant frequency to the rail axial load implies the capability. A numerical simulation in which a frequency load is applied at the railhead was performed. It has been validated that the resonant frequency can be captured by the horizontal loading. The influence of the disturbance in the periodicity on the resonant frequency has also been examined. Although the existence of the irregularity is not negligible in the aspect of the resonant frequency, its statistical behavior is predictable. Therefore the effect of the disturbance will be removed by utilizing this relation. It will be needed, in the future, to prove the theory discussed in this paper to be valid through experiments.

REFERENCES

Challenge H: For an even safer and more secure railway


